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DESIGN SENSITIVITY COEFFICIENTS FOR LINEAR ELASTIC BODIES WITH ZONES AND CORNERS BY THE DERIVATIVE BOUNDARY ELEMENT METHOD

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Abstract—The subject of this paper is the efficient and accurate determination of design sensitivity coefficients (DSCs) in an elastic solid with zones and corners. Direct differentiation of the relevant derivative boundary element method (DBEM) formulation of the problem is carried out here. Corners and zones in a body are treated carefully with conforming elements. A numerical implementation of the method is carried out with isoparametric quadratic boundary elements. The power of this approach is demonstrated through several numerical examples. It is shown that the DSCs of the various mechanical quantities of interest are obtained accurately and efficiently. In one example, even the DSC of the stress at a corner, with respect to the corner angle, is obtained very accurately by this method.

1. INTRODUCTION

Shape optimal design is an important topic in the structural design research area. Typically, the optimal shape of a 2-D or 3-D structural component is determined to minimize an objective function, subject to some constraints involving mechanical quantities such as displacements, tractions or stresses.

In almost all shape optimization processes, design sensitivity coefficients (DSCs), which are the rates of mechanical quantities with respect to a design variable, are essential for the determination of the optimum shape of the bodies. The design variable being considered here is a shape parameter that controls the shape of part or whole of the boundary of a body. The DSCs are then used as a guide to the best directions in nonlinear programming algorithms which typically iterate on the shape of the object along these directions until an optimal shape is obtained. Accurate and efficient determination of DSCs leads to a smaller number of iterations, thus leading to efficient design.

The subject of this paper is the accurate and efficient determination of design sensitivity coefficient (DSCs) for 2-D linear elasticity problems by the boundary element method (BEM). The bodies can have corners on their boundaries and can be divided into multiple zones.

The approach being used here is the direct analytical differentiation (DDA) of the governing boundary element method formulation of the problem. The exact differentiation eliminates errors that might occur from finite difference methods and leads to closed form integral equations for the desired sensitivities. These equations are then solved by numerical discretization. This approach is very accurate and efficient.

There are some papers in the literature that determine DSCs by DDA of the BEM formulation of a problem. Barone and Yang (1988) have used this approach for 2-D linear elasticity and have solved a simple example of an ellipse without corners. Rice and Mukherjee (1990) have solved DSCs for axisymmetric elasticity while Kane and Saigal (1988), Zhang and Mukherjee (1990), Saigal *et al.* (1989), Aithal *et al.* (1990) have solved planar, axisymmetric and some 3-D problems, respsectively. In Kane and Saigal (1988), Saigal *et al.* (1989) and Aithal *et al.* (1990), the authors first discretize the BEM equations and then differentiate them analytically with respect to shape variables, while in Barone and Yang (1988), Rice and Mukherjee (1990) and Zhang and Mukherjee (1989), the authors follow the opposite approach. It seems more appealing, intuitively, to start with differentiation of the relevant BEM equations and follow this by discretization. The two approaches do, as expected, eventually lead to the same equations.

Two difficult problems in this area of research are the accurate determination of sensitivities of boundary stresses and the determination of displacements, tractions, stresses and their sensitivities at corners on the boundary of a body. The finite element method (FEM) has problems with the determination of accurate boundary stresses while the approach of Barone and Yang (1988) requires the integration of strongly singular kernels in order to obtain these quantities. The problem of DSCs for stresses, at a point on a corner of the boundary of a body, has not been solved in an elegant manner before this work.

Ghosh *et al.* (1986, 1987), in two recent papers, have presented a BEM formulation in which the tractions and displacement derivatives are primary boundary variables. This formulation only has logarithmically singular kernels for 2-D problems. The boundary stresses can be obtained from the boundary values of tractions and displacement derivatives by purely algebraic calculations. While the original work (Ghosh *et al.*, 1986) used non-conforming elements at corners, the present work uses conforming elements, so that a source point can be placed directly at a corner.

The present work, which is based on DDA of the derivative BEM formulation (Ghosh *et al.*, 1986), makes two important contributions to the literature in this field. The first is the elegant and accurate determination of stress sensitivities at a regular point on the boundary of a body. The BEM formulation (Ghosh *et al.*, 1986) is analytically differentiated with respect to a shape parameter to yield an integral equation for the sensitivities of tractions and tangential derivatives of displacements on the boundary of a body. The new differentiated kernels are completely regular for 2-D problems! Then the boundary stress sensitivities are obtained directly as linear combinations of the sensitivities of tractions and tangential derivatives of displacements. The accuracy of stress sensitivities at a boundary point, therefore, is of the same order as that of the sensitivities of tractions and displacement derivatives.

The second important contribution is in the treatment of corners and zones. Conforming elements are used at corners in the present work. A source point is placed directly at a corner and the number of field quantities (tractions and displacement derivatives) is doubled at a corner since the components of these quantities are not necessarily continuous across it. Therefore, BEM equations cannot give enough information for solving the mechanical quantities. Extra equations at a corner come from the stress relations. In some special cases (for instance, a right angled corner), stress components are continuous at corners. This is not true in general. Stress discontinuities can occur at corners if the angle is not 90 degrees, even though the stress components are bounded there. These two different situations are carefully treated in this paper.

The above ideas have been implemented in a computer program for the determination of DSCs for planar elasticity problems. Numerical results are presented in this paper for DSCs for planar bodies without and with corners. A very interesting example is the determination of the sensitivity of stresses, at the tip of a wedge, with respect to the wedge angle. The numerical results reported here are generally very accurate.

2. A DBEM FORMULATION FOR PLANAR ELASTICITY

Ghosh *et al.* have recently proposed a derivative boundary element method (DBEM) formulation for linear elasticity in which the tractions and tangential derivatives of displacements (Ghosh *et al.*, 1986, for 2-D problems) or tractions and displacement gradients (Ghosh and Mukherjee, 1987, for 3-D problems) are the primary variables on the boundary of a body. An analogous formulation has also been presented by Okada *et al.* (1988).

The BEM equations for two-dimensional linear elasticity for a simply connected region B can be written as (Ghosh *et al.*, 1986):

$$\int_{\partial B} [U_{ij}(P,Q)\tau_i(Q) - W_{ij}(P,Q)\Delta_i(Q)]\,\mathrm{d}s(Q) = 0 \tag{1}$$

where U_{ij} is available in many references (e.g. Mukherjee, 1982) and, for plane strain

$$W_{ij} = \frac{1}{4\pi(1-\nu)}[2(1-\nu)\psi\delta_{ij} + \varepsilon_{ik}r_{,j}r_{,k} + (1-2\nu)\varepsilon_{ij}\ln r].$$

In this equation for W_{ij}, the source and field points are P and Q (with capital letters denoting points on the boundary ∂B and lower case letters denoting points inside B), r is the Euclidean distance between P and Q and ψ is the angle between the vector $r(P, Q)$ and a reference direction. Also, τ_i and Δ_i are the components of the traction and tangential derivative of the displacement ($\partial u_i/\partial s = \Delta_i$), respectively, with s the curvilinear coordinate measured along the boundary ∂B of the planar body. Finally, $\varepsilon_{11} = \varepsilon_{22} = 0, \varepsilon_{12} = -\varepsilon_{21} = 1$, ν is Poisson's ratio and δ_{ij} is the Kronecker delta. A comma following r denotes a derivative with respect to a field point coordinate. It is very important to note that W_{ij} has only a logarithmic singularity (same as U_{ij}) as r goes to zero. A constraint equation

$$\int_{\partial B_1} \Delta_i \, ds = u_i(2) - u_i(1)$$

(where ∂B_1 is a suitable part of ∂B with $u_i(1)$ and $u_i(2)$ the values of u_i at the beginning and end of ∂B_1) must be included for certain problems.

As can be seen from eqn (1), the traction and tangential displacement derivative vectors are the primary unknowns on ∂B in this formulation. It has been shown that the stress components at a regular point on ∂B, for plane strain, can be written in terms of the components of τ and Δ as (Sladek and Sladek, 1986; Cruse and Vanburen, 1971):

$$\sigma_{ij} = A_{ijk}\tau_k + B_{ijk}\Delta_k \tag{2}$$

where

$$A_{ijk} = (n_i n_j + c_1 t_i t_j)n_k + (n_i t_j + n_j t_i)t_k$$
$$B_{ijk} = c_2 t_i t_j t_k$$

with $c_1 = \nu/(1-\nu)$, $c_2 = 2G/(1-\nu)$ and G the shear modulus of the material. Also, n_i and t_i are the components of the unit (outward) normal and (counter-clockwise) tangential vectors at a point on ∂B. Thus, if τ and Δ are primitive variables on ∂B in a BEM formulation, then these quantities, as well as σ_{ij}, can be obtained on ∂B with very high accuracy.

The corresponding DBEM equation for the sensitivities are obtained by differentiating eqn (1) with respect to a shape design variable b (see Zhang and Mukherjee, 1989):

$$\int_{\partial B} [U_{ij}(\mathbf{b}, P, Q)\overset{*}{\tau}_i(\mathbf{b}, Q) - W_{ij}(\mathbf{b}, P, Q)\overset{*}{\Delta}_i(\mathbf{b}, Q)]\, ds(\mathbf{b}, Q)$$
$$+ \int_{\partial B} [\overset{*}{U}_{ij}(\mathbf{b}, P, Q)\tau_i(\mathbf{b}, Q) - \overset{*}{W}_{ij}(\mathbf{b}, P, Q)\Delta_i(\mathbf{b}, Q)]\, ds(\mathbf{b}, Q)$$
$$+ \int_{\partial B} [U_{ij}(\mathbf{b}, P, Q)\tau_i(\mathbf{b}, Q) - W_{ij}(\mathbf{b}, P, Q)\Delta_i(\mathbf{b}, Q)]\, d\overset{*}{s}(\mathbf{b}, Q) = 0 \tag{3}$$

where a superscript * denotes a derivative with respect to a typical component of $\mathbf{b}$. It has been shown (Barone and Yang, 1988) that

$$\overset{*}{U}_{ij}(\mathbf{b}, P, Q) = U_{ij,k}(\mathbf{b}, P, Q)[\overset{*}{x}_k(Q) - \overset{*}{x}_k(P)] \tag{4}$$

where, by virtue of the fact that

$$\overset{*}{x}_k(Q) - \overset{*}{x}_k(P) \sim O(r).$$

$\overset{*}{U}_{ij}$ is completely regular! A similar argument is used to show that $\overset{*}{W}_{ij}$ is also regular. Finally (Barone and Yang, 1988),

$$\mathrm{d}\overset{*}{s}(\mathbf{b}, Q) = [\overset{*}{x}_{k,k}(Q) - n_i n_j \overset{*}{x}_{i,j}(Q)]\,\mathrm{d}s(\mathbf{b}, Q). \tag{5}$$

A more convenient formula for $\mathrm{d}\overset{*}{s}$ is given in Appendix A. A very interesting feature of eqn (3) is that its first line is identical to eqn (1) with the sensitivities replacing the tractions and displacement derivatives. Analogous to the usual BEM problem, half of the sensitivities on ∂B must be prescribed and the rest can then be determined from eqn (3). Thus, the sensitivity problem has the same coefficient matrices as the original BEM problem with a known right hand side (since τ and Δ on ∂B are known at this stage). This known right hand side involves the evaluation of regular integrals which is very easy to perform accurately.

The equation for the sensitivity of stress at a regular point on ∂B is obtained by differentiating eqn (2) with respect to the design variable $\mathbf{b}$:

$$\overset{*}{\sigma}_{ij} = A_{ijk}\overset{*}{\tau}_k + B_{ijk}\overset{*}{\Delta}_k + \overset{*}{A}_{ijk}\tau_k + \overset{*}{B}_{ijk}\Delta_k. \tag{6}$$

The above equation expresses $\overset{*}{\sigma}_{ij}$ as a linear combination of τ_i, Δ_i and their sensitivities. Hence, one expects $\overset{*}{\sigma}_{ij}$ to be obtained as accurately as $\overset{*}{\tau}_i$ and $\overset{*}{\Delta}_i$.

3. CORNERS AND ZONES

3.1. *Corners*

The real solid body may include some corners across each of which there is a jump in the unit vectors **n** and **t** which are normal and tangential to the boundary ∂B. Consequently, discontinuities in both the tractions and tangential derivatives of displacements will occur at a corner. Therefore, eight quantities are of interest at a corner in the 2-D elasticity problem, only four of which are prescribed from the boundary conditions. If a source point P is placed exactly at a corner (conforming boundary elements) one obtains two BEM equations at P, but two more independent equations are still necessary. Fortunately, this information can be obtained by considering the behavior of stress components at a corner point.

3.1.1. *General corners: stress components are discontinuous at a corner.* A corner point on the boundary of the body can be viewed locally as the tip of a wedge, as indicated in Fig. 1. The wedge problem is a classical elasticity problem and has been investigated by many researchers for many years (Timoshenko and Goodier, 1970). The situation considered here is that the stress components are bounded at the tip as well as throughout the whole wedge, and that only distributed loads are applied on the wedge faces. Also, the assumption of

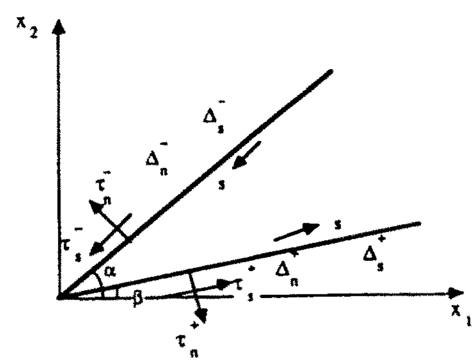

Fig. 1. The elasticity problem in a wedge.

uniform local boundary loads on the two faces of the wedge is made because each corner region is assumed to be small enough compared to the whole body.

A general stress function in polar coordinates, for planar elasticity problems, is available (Timoshenko and Goodier, 1970). The function chosen here is of the form

$$\phi(r,\theta) = r^2[A+B\theta+C\cos(2\theta)+D\sin(2\theta)] \tag{7}$$

because this function has four unknown constants, gives bounded stresses in the wedge and contains the only terms from the general solution that give constant tractions on the wedge faces. The corresponding stress components in polar coordinates are:

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2},\quad \sigma_\theta = \frac{\partial^2\phi}{\partial r^2},\quad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right). \tag{8}$$

Clearly, four coefficients A, B, C and D in the stress function have to be determined to satisfy the four boundary tractions which represent the local traction information around the corner.

Substituting the boundary conditions (Fig. 1)

$$\begin{aligned} &\text{at } \theta=\beta,\quad \sigma_\theta = \tau_n^+,\quad \sigma_{r\theta} = -\tau_s^+ \\ &\text{at } \theta=\alpha,\quad \sigma_\theta = \tau_n^-,\quad \sigma_{r\theta} = -\tau_s^- \end{aligned} \tag{9}$$

into eqn (7), one can obtain A, B, C and D easily (see Appendix B).

After lengthy algebra and use of the symbolic computer program MACSYMA, a rather simple solution is obtained for the stress jumps at the tip of the wedge. Writing, for simplicity, the stress jumps for the special case $\beta = 0$, one can show that

$$\begin{aligned} \Delta\sigma_{11} &= [\alpha+\sin\alpha\cos\alpha]f(\tau,\alpha) \\ \Delta\sigma_{22} &= [\alpha-\sin\alpha\cos\alpha]f(\tau,\alpha) \\ \Delta\sigma_{12} &= \sin^2\alpha f(\tau,\alpha) \end{aligned} \tag{10}$$

where

$$f(\tau,\alpha) = \frac{(\tau_n^+-\tau_n^-)+(\tau_s^++\tau_s^-)\tan\alpha}{\alpha-\tan\alpha}.$$

Here σ_{11}, σ_{12} and σ_{22} are components in the global Cartesian axes.

By observing eqn (10), it is obvious that the stress components are not, in general, continuous at the tip, even though they are bounded there. This is because there exists the term $Br^2\theta$ in the stress function and the coefficient B does not, in general, vanish. The special cases when the stress components have no jump at the tip will be discussed later in this paper.

It is easiest to understand the singular behavior of the displacement derivatives by considering the special case with only $B \neq 0$ in eqn (7). Now, once again, a general orientation of the corner ($\beta \neq 0$) is being considered. For this case, from eqn (8), the stress components in polar coordinates have the following form:

$$\sigma_r = 2B\theta,\quad \sigma_\theta = 2B\theta,\quad \sigma_{r\theta} = -B.$$

Substituting the above stresses into the constitutive relations for plane strain elasticity, one can obtain the Cartesian displacement components in terms of polar coordinate variables as the following:

$$u_1 = 2pr\theta\cos\theta + 4qr\ln r\sin\theta - kr\sin\theta + E$$

$$u_2 = 2pr\theta\sin\theta - 4qr\ln r\cos\theta + kr\cos\theta + F \quad (11)$$

where $p = B(1-2\nu)/2G$, $q = 2B(1-\nu)$ and E, F and k are integration constants. Here, E and F represent rigid body translations while k represents the rigid body rotation.

Evaluating eqn (11) along the boundary of the wedge and taking derivatives with respect to the curvilinear coordinate s, one can easily get the tangential displacement derivatives Δ_1 and Δ_2 as:

$$\text{at } \theta = \beta, \quad \Delta_1 = 2p\beta\cos\beta + 4q\ln r\sin\beta + 4q\sin\beta - kr\sin\beta$$

$$\Delta_2 = 2p\beta\sin\beta - 4q\ln r\cos\beta - 4q\cos\beta + kr\cos\beta$$

$$\text{at } \theta = \alpha, \quad \Delta_1 = -2p\alpha\cos\alpha - 4q\ln r\sin\alpha - 4q\sin\alpha + kr\sin\alpha$$

$$\Delta_2 = -2p\alpha\sin\alpha + 4q\ln r\cos\alpha + 4q\cos\alpha - kr\cos\alpha \quad (12)$$

Note that Δ_1 and Δ_2 have $\ln r$ singularities at the tip of the wedge. This is directly induced by the $Br^2\theta$ term. Therefore, the $Br^2\theta$ term in the stress function causes not only the discontinuity in stresses but also the singularity in displacement derivatives.

Further analysis of eqn (2) reveals that the boundary stress components only involve the tangential component of Δ. Since, by assumption, σ is bounded at a corner, so are Δ_s on either side of it. It can be shown from eqn (12) that Δ_n^- and Δ_n^+, associated with rigid body rotation, become singular in this case.

A BEM implementation of the general corner is described next. Suppose, for clarity, that the tractions are prescribed on either side of a general corner. Mixed boundary conditions can be taken care of by a modification of the following.

The first step is to obtain a general solution for Δ_1 and Δ_2, on either side of a corner, in terms of the prescribed tractions and the (unknown) rigid body rotation k. For this, Appendix B would be useful. The result is the expressions (12), corresponding to B (assuming $B \neq 0$), as well as other bounded terms from A, C and D of eqn (7). If, in a given example, B happens to vanish, then one has a special corner, with continuous stress and bounded Δ. This is discussed in the next section. Otherwise, the existence of the $\ln r$ singularity in Δ_1 and Δ_2, on either side of a general corner, requires special attention in a BEM implementation. Now a special pair of (small, straight) segments ∂B_s (Fig. 2) are placed next to a corner point and a source point P is placed at the corner O as usual. Solution (12), which originates from the $Br^2\theta$ term in eqn (7), is valid locally if the segments are small. The other terms in $\phi(r, \theta)$ from eqn (7) give bounded Δ and must also be included in general.

The DBEM eqn (1) is separated into two parts, one over ∂B_s and the other over the remainder $\partial B - \partial B_s$:

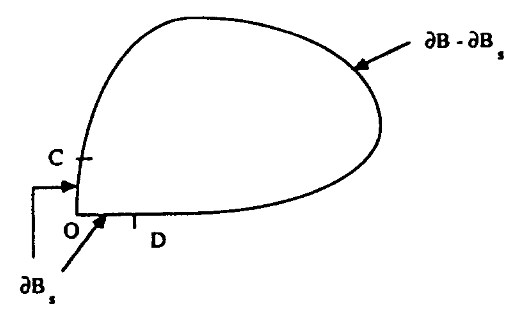

Fig. 2. Special segments on the boundary.

$$\int_{\partial B-\partial B_s}[U_{ij}(P,Q)\tau_i(Q)-W_{ij}(P,Q)\Delta_i(Q)]\,ds$$
$$+\int_{\partial B_s}[U_{ij}(P,Q)\tau_i(Q)-W_{ij}(P,Q)\Delta_i(Q)]\,ds=0. \quad (13)$$

The expressions in eqn (12) (together with other terms from A, C and D) are integrated exactly, even for the worst case when the source point is at the corner itself and one term in $W_{ij}\Delta_i$ has the form $(\ln r)^2$. The local rotation k, at the corner, is unknown in eqn (12), but can be solved once the global BEM equations, over the entire ∂B, are assembled. In effect, the quantity k assures the compatibility of the local solution at a corner and the global solution of a problem.

The sensitivity equations at a general corner must be obtained next. If $\overset{*}{\tau}$ractions are prescribed at a general corner, their sensitivities vanish. The sensitivities $\overset{*}{\Delta}_1$, $\overset{*}{\Delta}_2$ on either side of the corner, are obtained by differentiating the appropriate expressions for Δ with respect to the design variables. This leads to an unknown quantity $\overset{*}{k}$ which is solved from the differentiated form of eqn (13). In this way, the sensitivities on the boundary ∂B, of a body with general corners, can be obtained.

3.1.2. *Special corners: stress components are continuous at special corners.* Looking at the entire problem from a mathematical as well as a physical point of view, it is seen that the following simple situations lead to continuity of stresses (and bounded Δ) at corners. Other situations with continuous σ are also possible.

A right angled corner with arbitrary applied tractions.
An acute angled unloaded corner ($\tau_n^+=\tau_n^-=\tau_s^+=\tau_s^-=0$) where $\sigma_{ij}=0$ (Williams, 1952).
A corner which arises from using symmetry or zoning in a problem where the point was originally regular or a special corner.

Once the stresses are continuous around a corner, the following equations hold from eqn (2):

$$A_{ijk}^-\tau_k^-+B_{ijk}^-\Delta_k^-=A_{ijk}^+\tau_k^++B_{ijk}^+\Delta_k^+\quad(i,j,k=1,2). \quad (14)$$

The above gives three equations, of which at least two are linearly independent. Therefore, the BEM eqns (1) plus eqns (14) give enough equations for solving the boundary unknowns, including four from each corner.

This global system is overdetermined since extra equations arise from the stress relations (14). The system, however, has full column rank, is consistent and the number of linearly independent equations equals the number of unknowns. Regular QR decomposition is used to solve this system.

The corresponding sensitivity equations are obtained by differentiating eqn (14) with respect to the design variable b. The expression is the following:

$$\overset{*}{A}{}_{ijk}^-\tau_k^-+\overset{*}{B}{}_{ijk}^-\Delta_k^-+A_{ijk}^-\overset{*}{\tau}{}_k^-+B_{ijk}^-\overset{*}{\Delta}{}_k^-=\overset{*}{A}{}_{ijk}^+\tau_k^++\overset{*}{B}{}_{ijk}^+\Delta_k^++A_{ijk}^+\overset{*}{\tau}{}_k^++B_{ijk}^+\overset{*}{\Delta}{}_k^+. \quad (15)$$

These corner sensitivity equations, together with eqn (3), can be used to solve for the unprescribed boundary sensitivities in a body with special corners.

3.2. *Zones*

Multiple zones have been treated in a BEM program by several authors (e.g. Liggett and Liu, 1983). Here a consistent approach is presented for the treatment of zones and corners in a DBEM program.

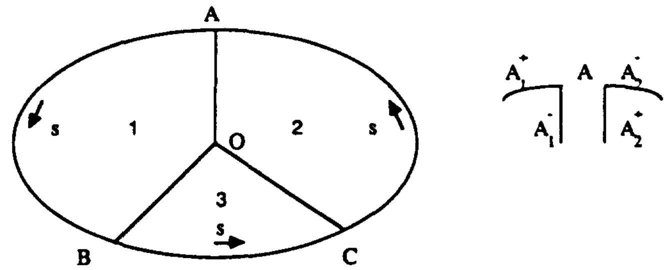

Fig. 3. Zones.

A 2-D situation is shown in Fig. 3 with three zones. A three-zone problem is considered here for simplicity, with the total body having a smooth boundary. The material in each zone is considered to be homogeneous, isotropic and linearly elastic, but the elastic properties can vary from one zone to another. The idea of modeling the three-zone problem, which is easy to extend to a body with an arbitrary number of zones, is outlined below. This algorithm works fine if the stresses are bounded throughout the composite body. This is expected in a body with uniform material properties, which is zoned for improving the efficiency and accuracy of BEM modeling (Rudolphi, 1983). The general problem with different materials often has stress singularities at points where the zones meet (Liggett and Liu, 1983).

The basic idea is to treat the body as three separate bodies. Each body (zone) has an external boundary, an internal boundary and three corner points (stresses are continuous at these points because they are generated, by the zoning process, from a body with a smooth boundary). Let the boundaries of the three bodies be discretized. Interboundary compatibility at the interface nodes must be satisfied. Conforming elements are used at each corner.

Assume there are N_{ii} *regular* exterior points for each body ($i = 1, 2, 3$ for this example) and N_{ij} *regular* interface points between zones (i.e. N_{11} regular exterior points for zone 1 and N_{12} regular interface points between zones 1 and 2, etc.). A regular point is one such that the boundary of zone i, on which it lies, is locally smooth there.

There are four kinds of equations or relations to be considered.

3.2.1. *The BEM equations*. BEM equations are used in each zone and there are two equations for each point (regular or corner point). For example, there are $2(N_{11}+N_{12}+N_{13})+6$ BEM equations for body 1. The total number of equations is

$$2\sum_{i=1}^{3}\sum_{j=1}^{3} N_{ij}+18. \tag{16}$$

3.2.2. *The external boundary conditions*. Two quantities are specified at each regular external point. These quantities can, in general, jump at a corner point. Thus, two quantities are specified on each side of the corner points. The total number of external boundary conditions is:

$$2\sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij}N_{ij}+12 \tag{17}$$

where δ_{ij} is the Kronecker delta.

3.2.3. *The interface conditions*. At each interface point (say between A_1^- and A_2^+), one has the equations ($i, j = 1, 2, 3, i \neq j$):

$$\Delta_1^{(i)} = -\Delta_1^{(j)}, \quad \Delta_2^{(i)} = -\Delta_2^{(j)},$$
$$\tau_1^{(i)} = -\tau_1^{(j)}, \quad \tau_2^{(i)} = -\tau_2^{(j)}.$$

The total number of interface equations is

$$2\sum_{i=1}^{3}\sum_{j=1}^{3}(1-\delta_{ij})N_{ij}+24. \tag{18}$$

3.2.4. *The corner equations*. Corner equations have been discussed in detail in Section 3.1 of this paper. There are nine corners in this example and three equations for each corner [see eqns (14) and (15)]. A corner equation is applied between points in one zone, say between A_1^- and A_1^+. Therefore, the total number of equations is 27.

Adding the terms in Sections 3.2.1–3.2.4 together, one obtains the total number of equations as:

$$4\sum_{i=1}^{3}\sum_{j=1}^{3}N_{ij}+81.$$

Since there are four primary quantities for each regular point and eight primary quantities for each corner point, the total number of unknowns is:

$$4\sum_{i=1}^{3}\sum_{j=1}^{3}N_{ij}+72.$$

The system is overdetermined because of the stress continuity equations at corners. However, the equations are consistent and sufficient and can be solved to get a unique solution.

4. NUMERICAL IMPLEMENTATION

4.1. *Discretization of equations—I: stresses are continuous at corners of the body*

The BEM equations (1) (for tractions and tangential displacement derivatives) and (3) (for their sensitivities) are discretized in the usual way. The boundary ∂B is subdivided into piecewise quadratic, conforming boundary elements. The variables τ_i and Δ_i are assumed to be piecewise quadratic on these boundary elements. The logarithmically singular kernels are integrated by using log-weighted Gaussian integration. When special corners exist, the corner equations are added to the usual BEM equations, and all the equations are assembled together. The resulting systems are of the form

$$[A]\{\tau\}+[B]\{\Delta\} = \{0\} \tag{19}$$

$$[A]\{\mathring{\tau}\}+[B]\{\mathring{\Delta}\} = \{h\} \tag{20}$$

and after switching appropriate columns one obtains, for the unknowns $\{x\}$ and $\{\mathring{x}\}$ on the boundary

$$[K]\{x\} = \{r_1\} \tag{21}$$

$$[K]\{\mathring{x}\} = \{r_2\}. \tag{22}$$

Two points deserve mention here. First, eqns (21) and (22) have the same stiffness matrix $[K]$. The vector $\{r_2\}$ contains the contributions from the second and third lines of eqn (3). Second, eqns (21) and (22) are overdetermined but have full column rank. They have been solved by QR decomposition in the numerical examples that follow (Golub and Van Loan, 1989).

4.2. *Discretization of equations—II: stresses are discontinuous at corners of the body*

The modified BEM equations (13) are discretized in the usual way only on the boundary $\partial B-\partial B_s$. A pair of special segments ∂B_s is placed near each corner where the stresses are discontinuous. The analytical solutions for Δ_i, discussed earlier, are assumed to be valid in the segments if they are small enough. The first integral in (13) yields coefficient matrices for the nodal unknowns on the boundary $\partial B - \partial B_s$. The second integral in (13) yields the coefficient matrix for k and a known column from integrations of known functions such as the first three terms of each of (12). The singular function $\ln^2 r$ is integrated analytically. The resulting systems are of the form

$$[A]\{\tau\}+[B]\{\Delta\}+\{P\}k+\{f_1\} = 0 \tag{23}$$

and after switching appropriate columns one obtains, for the unknowns $\{x\}$ including Δ, τ, and k

$$[K]\{x\} = \{q_1\}. \tag{24}$$

If there are some special corners on the boundary, the corner equations for them are added to the above equations and the assembled system still has the same form. The system, then, becomes overdetermined and is solved by QR decomposition.

The corresponding sensitivity equations are obtained in analogous fashion. This system has the form

$$[K]\{\dot{x}\} = \{q_2\}. \tag{25}$$

4.3. *Numerical results*

All the numerical results discussed below are for plane strain with Poisson's ratio $\nu = 0.3$. The mechanical quantities τ, Δ and σ and their sensitivities are determined for each problem. The first two examples include special corners. A general corner is studied in the third problem. The last problem has two zones.

Example 1. A wedge of angle α subjected to tractions is shown in Fig. 4. These tractions are obtained from the stress function

$$\phi(r,\theta) = Ar^2 \sin 2\theta$$

with $A = -1/2 \sin 2\alpha$. In this special problem, the stress tensor σ is continuous at the tip of the wedge O (note that here the tractions are functions of α). The wedge angle α is the design variable in this problem. The analytical solution of this problem is

$$\sigma_{11} = \sigma_{22} = 0, \quad \sigma_{12} = 1/\sin 2\alpha$$

so that

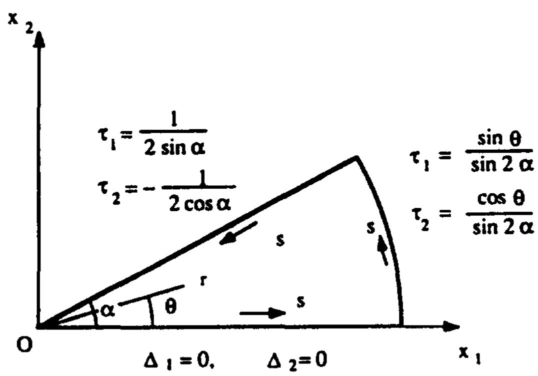

Fig. 4. Example 1 : special corner problem.

Table 1. Stress components and their sensitivities at the tip of the wedge (example 1)

	σ_{11}	σ_{22}	σ_{12}
Analytical σ_{ij}	0.0000000	0.0000000	1.1547005
Numerical σ_{ij}	-0.2115977E-5	0.1675500E-6	1.1546976
% Error	/	/	0.00025
Analytical σ^*_{ij}	0.0000000	0.0000000	-1.3333333
Numerical σ^*_{ij}	0.2448821E-4	0.1550109E-4	-1.3333032
% Error	/	/	0.0030

$$\overset{*}{\sigma}_{11} = \overset{*}{\sigma}_{22} = 0, \quad \overset{*}{\sigma}_{12} = \frac{\partial \sigma_{12}}{\partial \alpha} = -\frac{2\cos 2\alpha}{\sin^2 2\alpha}.$$

This example provides an opportunity to test the present method for the determination of the stress at the tip of a wedge and its sensitivity with respect to the wedge angle. In this example (Appendix A),

$$\frac{\mathrm{d}\overset{*}{s}}{\mathrm{d}s} = \frac{1}{\alpha}$$

on the curved surface and $\mathrm{d}\overset{*}{s} = 0$ on the straight faces which undergo rigid body rotation.

The numerical results for the stress components and their sensitivities at O are given in Table 1. It is quite remarkable that the numerical result for $\overset{*}{\sigma}_{12}$ captures five significant digits of the analytical solution with only three quadratic boundary elements used to model the boundary ∂B!

Example 2. The classical problem of a body with an elliptical hole is considered in this example. Only a quarter of the ellipse needs to be modeled because of symmetry (Fig. 5). Here, $a = 2$, $b = 1$, $L = 30$, $\sigma_\infty = 1.0$. The corners here arise due to the use of symmetry of the problem. Hence they are special corners where the stresses are continuous. The semi-major axis a is the design variable in this problem. The analytical solution for the tangential

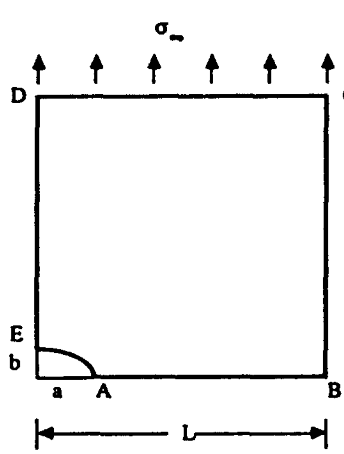

Fig. 5. Example 2: a body with an elliptical hole.

"skin" stress and its sensitivity on the ellipse (for an elliptical hole in an infinite plate) are (Barone and Yang, 1988):

$$\sigma_\theta = \sigma_x \frac{1+2m-m^2+2\cos 2\phi}{1+m^2+2m\cos 2\phi}$$

$$\overset{*}{\sigma}_\theta = -\sigma_x \frac{(1-m)(1+m^2+2m\cos 2\phi)-(m+\cos 2\phi)(1+2m-m^2+2\cos 2\phi)}{(1+m^2+2m\cos 2\phi)^2}\left(\frac{2b}{(a+b)^2}\right)$$

where ϕ is the eccentric angle and $m = (b-a)/(a+b)$.

The comparisons of analytical and numerical results for σ_θ and $\overset{*}{\sigma}_\theta$ are shown in Fig. 6a and b respectively. A total of 54 quadratic elements (20 elements are spaced at equal

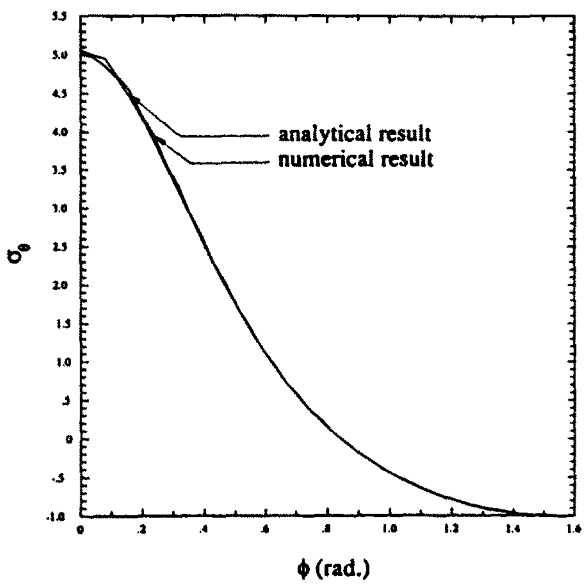

Fig. 6a. Angular variation of σ_θ around the quarter ellipse.

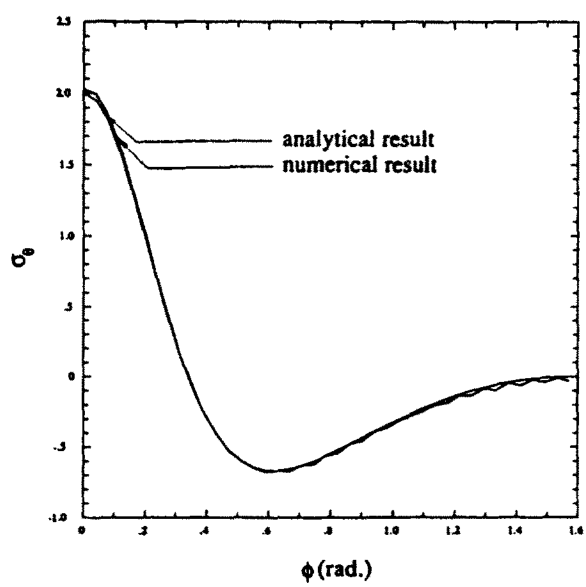

Fig. 6b. Angular variation of $\overset{*}{\sigma}_\theta$ around the quarter ellipse.

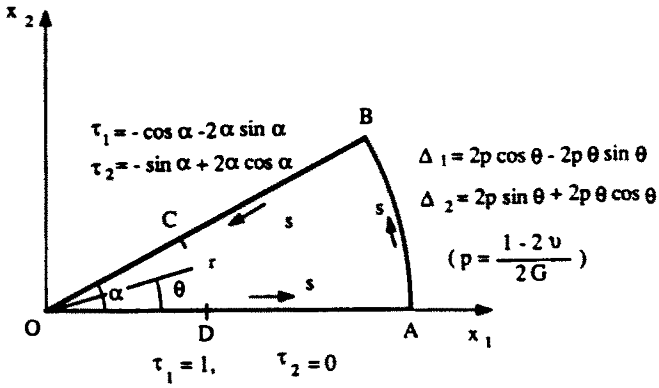

Fig. 7. Example 3: general corner problem.

increments of the eccentric angle on the quarter ellipse, 12 elements applied on AB, 14 on DE, four on BC and four on CD, respectively) are used for these numerical results. The density of elements on AB and ED is nonuniform, with small elements being placed near the points A and E, respectively. Problems involving stress concentrations are typically sensitive to the mesh around the stress concentration points. The mesh used here is the result of a limited convergence study and previous experience with such problems.

The results from the present method are seen to be very accurate over the entire region. In these figures, the numerical solutions, except for some very small oscillations, essentially agree with the analytical solutions within plotting accuracy. It is remarkable that the computed sensitivity of the stress concentration factor at A is 2.03 and the relative error is 1.54% compared to the analytical result of 2.0.

Example 3. The effect of corners where the stresses are discontinuous is studied by considering the problem of a wedge of angle α subjected to the tractions shown in Fig. 7. This solution is obtained from the stress function

$$\phi(r, \theta) = Br^2\theta$$

with $B = 1$. In this problem, the stress tensor σ is discontinuous at the tip of the wedge O. The design variable is still the angle α. The strategy discussed in Sections 3.1.1 and 4.2 is used here with $\partial B_s = COD$. A comparison of analytical and numerical results for $\dot{\Delta}_1$ and $\dot{\Delta}_2$ on the boundary CB are shown in Fig. 8a and b, respectively. A total of six quadratic

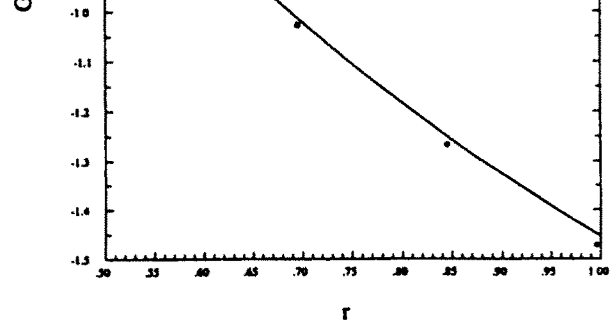

Fig. 8a. Variation of $G\dot{\Delta}_1$ along CB.

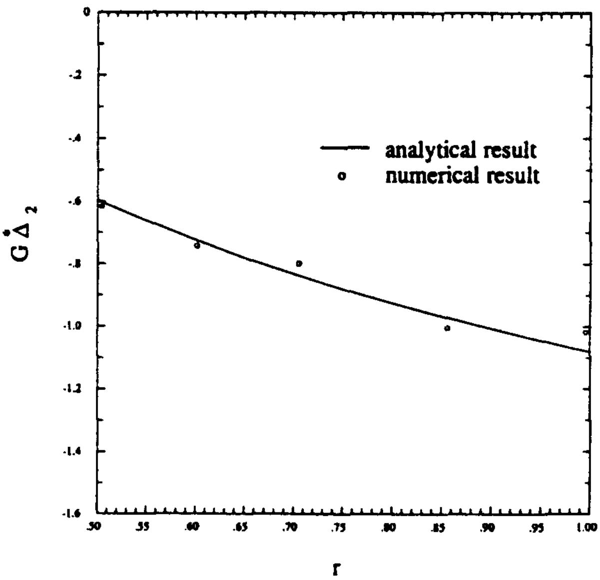

Fig. 8b. Variation of $G\dot{\Delta}_2$ along CB.

elements are used over $\partial B - \partial B_s$, two for each edge. All the numerical results lie close to the curves. The results are accurate. For more general problems, the special segments ∂B_s and the BEM elements near the transition points C and D must be chosen properly.

Example 4. The final problem considered in this paper is the determination of mechanical quantities and their sensitivities in the region are shown in Fig. 9. Two zones, with uniform material properties, are included in this problem. The applied boundary conditions in this problem are the same as those in example 1 (Fig. 4) and the analytical solution for this problem is the same as that from example 1. Numerical results for the stress components and their sensitivities at O are shown in Table 2. These are as accurate as those in Table 1 for example 1.

5. CONCLUDING REMARKS

The power of the DDA of the relevant DBEM equations, for the determination of the DSCs of an elastic problem, has been demonstrated in this paper. This approach, which uses tractions, tangential displacement derivatives, and their sensitivities as primary boundary variables, is a natural for the accurate determination of stress sensitivities on the boundary of a body. Boundary stress sensitivities, which are typically not easy to obtain accurately by numerical methods, have been obtained very accurately here, at regular as well as at corner points on the boundary of a body.

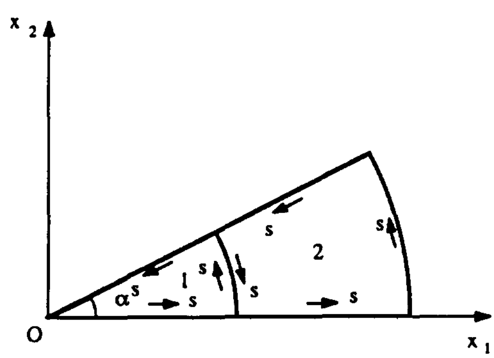

Fig. 9. Example 4: zones.

Table 2. Stress components and their sensitivities at the tip of the wedge (example 4)

	σ_{11}	σ_{22}	σ_{12}
Analytical σ_{ij}	0.0000000	0.0000000	1.1547005
Numerical σ_{ij}	-0.2432946E-5	-0.1914389E-5	1.1546984
Error (%)	/	/	0.00025
Analytical σ^*_{ij}	0.0000000	0.0000000	-1.3333333
Numerical σ^*_{ij}	0.4803036E-5	0.6862585E-5	-1.3333177
Error (%)	/	/	0.0012

The chosen numerical examples have analytical solutions available and serve as benchmark problems for testing the accuracy of the numerical algorithm. Of course, the computer program that has been generated here can be applied to carry out general sensitivity analysis of 2-D elastic problems.

Extensions of this work could be carried out to calculate DSCs of 3-D elastic problems (following the DBEM formulation of Ghosh and Mukherjee, 1987) as well as nonlinear problems with both material as well as geometrical nonlinearities. Some of the mathematical formulations for nonlinear problems have already been completed (Mukherjee and Chandra, 1989, 1990) and numerical implementations are under way.

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APPENDIX A

Consider the parametric equations for a curve

$$x_1 = f_1(\mathbf{b}, \eta)$$
$$x_2 = f_2(\mathbf{b}, \eta)$$

with $\eta \in [c, d]$ where $c, d \in R^{(1)}$. Now, one has a smooth mapping $I \times R^{(n)} \to R^{(2)}$. Here $\mathbf{b} \in R^{(n)}$, $(x_1, x_2) \in R^{(2)}$ and η is not a function of $\mathbf{b}$.

Now,

$$\mathrm{d}s = \left[\left(\frac{\partial x_1}{\partial \eta}\right)^2 + \left(\frac{\partial x_2}{\partial \eta}\right)^2\right]^{1/2} \mathrm{d}\eta$$

$$\mathrm{d}\overset{*}{s} = \frac{\partial}{\partial b}\left[\left(\frac{\partial x_1}{\partial \eta}\right)^2 + \left(\frac{\partial x_2}{\partial \eta}\right)^2\right]^{1/2} \mathrm{d}\eta$$

for a typical component of $\mathbf{b}$.

Thus,

$$\frac{\mathrm{d}\overset{*}{s}}{\mathrm{d}s} = \frac{\dfrac{\partial}{\partial b}\left|\dfrac{\partial \mathbf{x}}{\partial \eta}\right|}{\left|\dfrac{\partial \mathbf{x}}{\partial \eta}\right|}$$

(Chen, 1989).

As an example, consider the curved part of the boundary of the wedge in Fig. 4. Let

$$x_1 = R\cos(\eta\alpha), \quad x_2 = R\sin(\eta\alpha)$$

where $\eta \in [0, 1]$ and $b = \alpha$. Now

$$\left|\frac{\partial \mathbf{x}}{\partial \eta}\right| = R\alpha, \quad \frac{\partial}{\partial b}\left|\frac{\partial \mathbf{x}}{\partial \eta}\right| = R$$

so that

$$\frac{\mathrm{d}\overset{*}{s}}{\mathrm{d}s} = \frac{1}{\alpha}.$$

APPENDIX B

The coefficients A, B, C and D of eqn (7) are the following (see Fig. 1):

$$A = [-(\sin 2\gamma + 2\alpha\cos 2\gamma - 2\beta)\tau_s^+ - (2\alpha\sin 2\gamma - \cos 2\gamma + 1)\tau_n^+ + (\sin 2\gamma - 2\beta\cos 2\gamma + 2\alpha)\tau_s^- + (2\beta\sin 2\gamma + \cos 2\gamma - 1)\tau_n^-]/4t$$

$$B = [-(1 - \cos 2\gamma)\tau_s^+ - (1 - \cos 2\gamma)\tau_s^- + \sin 2\gamma(\tau_n^+ - \tau_n^-)]/2t$$

$$C = [-(\sin 2\alpha - \sin 2\beta + 2\gamma\cos 2\alpha)\tau_s^+ + (\sin 2\alpha - \sin 2\beta + 2\gamma\cos 2\beta)\tau_s^- + (\cos 2\alpha - \cos 2\beta)(\tau_n^+ - \tau_n^-)]/4t$$

$$D = [(\cos 2\alpha - \cos 2\beta - 2\gamma\sin 2\alpha)\tau_s^+ - (\cos 2\alpha - \cos 2\beta - 2\gamma\sin 2\beta)\tau_s^- + (\sin 2\alpha - \sin 2\beta)(\tau_n^+ - \tau_n^-)]/4t$$

where

$$\gamma = \beta - \alpha$$
$$t = \gamma\sin 2\gamma + \cos 2\gamma - 1.$$